

# Topological Entropy – A Survey

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## 1 Introduction

The aim of this paper is to provide an overview of the (in our opinion) most important results on topological entropy, starting from the original definition by Adler, Konheim, and McAndrew in 1965 up to nowadays. We cite the corresponding papers in chronological order and restate the theorems they contain trying to use a unified notation. Since the theory of topological entropy is closely related to that of metric entropy via the variational principle, we also cite some results on metric entropy, which are useful for the computation or estimation of topological entropy. Our interest is mainly in results which either concern elementary properties or (preferably general) inequalities or formulas for topological entropy. Hence, this survey excludes large parts of the entropy theory, as e.g., all the work concerning topological entropy of geodesic flows and relations to curvature, relations between topological entropy and chaos, entropy of symbolic systems, symbolic extension entropy, and many papers dealing with partial results aiming at Shub's entropy conjecture. To get an overview of the history of entropy in dynamical systems, we highly recommend the excellent survey of Anatole Katok [34]. The same author also wrote a survey paper on Shub's entropy conjecture, see [32]. Another highly recommended survey on entropy with an emphasis on relations between entropy, Lyapunov exponents and dimension, written by Lai-Sang Young, can be found under the URL <http://www.math.nyu.edu/~lsy/papers/entropy.pdf>, or in [23, Ch. 16]. Further references for the theory of metric and topological entropy are the books by Alsedà, Llibre, Misiurewicz [2], Katok and Hasselblatt [35], Mañé [44], Pollicott [60], Robinson [63], Walters [68], and Downarowicz [17]. Finally, we want to remark that this paper has not been peer-reviewed and hence should not be used as a reference in scientific work. None of the results but all of the mistakes are due to the author.

## 2 The Sixties

### 2.1 Topological Entropy (1965)

In [1], Adler, Konheim and McAndrew introduce the topological entropy of a continuous map  $f : X \rightarrow X$  on a compact topological space  $X$ :

#### 2.1 Definition:

Let  $N(\mathcal{U})$  denote the minimal cardinality of a subcover of an open cover  $\mathcal{U}$  of  $X$ , and let  $H(\mathcal{U}) := \log N(\mathcal{U})$ .<sup>1</sup> Define the join of open covers  $\mathcal{U}_1, \dots, \mathcal{U}_n$

<sup>1</sup>Usually, one takes the logarithm to the base 2. But the choice of the base is not essential, since its change only results in a constant scaling factor.

by

$$\bigvee_{i=1}^n \mathcal{U}_i := \{U_1 \cap \dots \cap U_n : U_i \in \mathcal{U}_i, i = 1, \dots, n\},$$

and let

$$f^{-1}(\mathcal{U}) := \{f^{-1}(U) : U \in \mathcal{U}\}$$

for any open cover  $\mathcal{U}$  of  $X$ . Define the topological entropy of  $f$  with respect to an open cover  $\mathcal{U}$  by<sup>2</sup>

$$h_{\text{top}}(f, \mathcal{U}) := \lim_{n \rightarrow \infty} \frac{1}{n} H \left( \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U}) \right),$$

and the topological entropy of  $f$  by

$$h_{\text{top}}(f) := \sup_{\mathcal{U}} h_{\text{top}}(f, \mathcal{U}),$$

where the supremum is taken over all open covers  $\mathcal{U}$  of  $X$ .

Obviously,  $h_{\text{top}}(f) \in [0, \infty]$ . The following properties of  $h_{\text{top}}(f)$  are shown:

**2.2 Theorem:**

(i) Topologically conjugate maps have the same topological entropy:

$$h_{\text{top}}(f) = h_{\text{top}}(h \circ f \circ h^{-1}),$$

where  $h$  is a homeomorphism.

(ii) For the topological entropy of iterates the following formula holds:

$$h_{\text{top}}(f^k) = k \cdot h_{\text{top}}(f) \text{ for all } k \in \mathbb{N}.$$

(iii) If  $f$  is a homeomorphism, then

$$h_{\text{top}}(f^k) = |k| \cdot h_{\text{top}}(f) \text{ for all } k \in \mathbb{Z}.$$

(iv) Assume that  $X$  and  $Y$  are compact topological spaces and  $f : X \circlearrowleft$  and  $g : Y \circlearrowleft$  are continuous maps. Then

$$h_{\text{top}}(f \times g) \leq h_{\text{top}}(f) + h_{\text{top}}(g).$$

If, in addition,  $X$  and  $Y$  are Hausdorff,<sup>3</sup> then

$$h_{\text{top}}(f \times g) = h_{\text{top}}(f) + h_{\text{top}}(g).$$

<sup>2</sup>Note that the limit in the definition of  $h_{\text{top}}(f, \mathcal{U})$  exists because of subadditivity.

<sup>3</sup>In the original paper, it was claimed that equality holds without the additional property of  $X$  and  $Y$  being Hausdorff. But in [22] ("The Product Theorem for Topological Entropy"), Goodwyn showed that the proof of Adler, Konheim and McAndrew does not work without this assumption.

- (v) Let  $X_1, X_2 \subset X$  be two closed subsets of  $X$  with  $X_1 \cup X_2 = X$  and  $f(X_i) \subset X_i, i = 1, 2$ . Then

$$h_{\text{top}}(f) = \max \{h_{\text{top}}(f|_{X_1}), h_{\text{top}}(f|_{X_2})\}.$$

- (vi) Let  $\sim$  be an equivalence relation on  $X$  respected by  $f$ , and let  $\tilde{f}$  be the quotient map on  $X/\sim$ . Then

$$h_{\text{top}}(\tilde{f}) \leq h_{\text{top}}(f).$$

- (vii) If  $X$  is a compact metric space, then

$$h_{\text{top}}(f, \mathcal{U}_n) \rightarrow h_{\text{top}}(f),$$

if  $(\mathcal{U}_n)_{n \geq 1}$  is a sequence of open covers whose diameters converge to zero and such that  $\mathcal{U}_{n+1}$  is a refinement of  $\mathcal{U}_n$  for all  $n \in \mathbb{N}$ .

### 2.3 Remark:

Item (vi) can be restated by saying that if two maps  $f : X \curvearrowright$  and  $g : Y \curvearrowright$  satisfy the semiconjugacy identity  $h \circ f = g \circ h$  with a continuous surjection  $h : X \rightarrow Y$ , then  $h_{\text{top}}(g) \leq h_{\text{top}}(f)$ .

## 2.2 On the Topological Entropy of a Dynamical System (1969)

In [30], Ito proves the following result, which was formulated as a conjecture by Adler, Konheim and McAndrew [1]:

### 2.4 Theorem:

Let  $\varphi : \mathbb{R} \times X \rightarrow X, (t, x) \mapsto \varphi_t(x)$ , be a continuous flow on a compact metric space  $X$ . Then

$$h_{\text{top}}(\varphi_t) = |t| \cdot h_{\text{top}}(\varphi_1) \text{ for all } t \in \mathbb{R}.$$

## 2.3 Topological Entropy bounds Measure-Theoretic Entropy (1969)

In [21], Goodwyn proves the following theorem, which later was improved by the *variational principle* (already conjectured in Adler, Konheim and McAndrew [1]). The latter states that the topological entropy equals the supremum over the metric entropies with respect to all invariant Borel probability measures:

### 2.5 Theorem:

Let  $X$  be a compact metric space,  $f : X \curvearrowright$  a continuous map, and  $\mu$  an  $f$ -invariant Borel probability measure on  $X$ . Then

$$h_{\mu}(f) \leq h_{\text{top}}(f),$$

i.e., the metric entropy is bounded from above by the topological entropy.

### 3 The Seventies

#### 3.1 Topological Entropy and Axiom A (1970)

In [7], Bowen proves a couple of important results:

##### 3.1 Theorem:

- (i) Let  $X$  be a compact metric space and  $f : X \rightarrow X$  continuous. Then

$$h_{\text{top}}(f) = h_{\text{top}}(f|_{\Omega(f)}),$$

where  $\Omega(f)$  denotes the non-wandering set<sup>4</sup> of  $f$ .

- (ii) If  $f$  is a diffeomorphism on a compact manifold which satisfies Axiom A<sup>5</sup>, then  $f$  has a neighborhood  $N_f$  in  $\text{Diff}(M)$  such that  $h_{\text{top}}(f) \leq h_{\text{top}}(g)$  for all  $g \in N_f$ .
- (iii) Let  $f$  be an expansive homeomorphism<sup>6</sup> of a compact metric space. Then

$$h_{\text{top}}(f) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f),$$

where  $P_n(f)$  is the number of fixed points of  $f^n$ .

- (iv) Let  $f$  be a  $C^1$ -diffeomorphism (or  $C^1$ -map) on a smooth compact manifold satisfying Axiom A. Then

$$h_{\text{top}}(f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f)$$

and  $h_{\text{top}}(f) > 0$  unless  $\Omega(f)$  is finite.

##### 3.2 Remark:

A particularly short proof of (i) can be found in the book Alsedà, Llibre, Misiurewicz [2].

#### 3.2 An Estimate from above for the Entropy and the Topological Entropy of a $C^1$ -diffeomorphism (1970)

In [31], Ito proves the following theorem, which gives an upper estimate of the topological entropy of a diffeomorphism:

<sup>4</sup>A point  $x \in X$  is said to be *non-wandering* with respect to  $f$  if for every neighborhood  $U$  of  $x$  there is  $n \geq 1$  such that  $f^n(U) \cap U \neq \emptyset$ . The non-wandering set of  $f$  is the set of all non-wandering points.

<sup>5</sup> $f$  satisfies Axiom A if the non-wandering set  $\Omega(f)$  is hyperbolic and the set of periodic points is dense in  $\Omega(f)$ .

<sup>6</sup>That means, there is  $\delta > 0$  such that  $\sup_{-\infty < i < \infty} d(f^i(x), f^i(y)) > \delta$  if  $x \neq y$ .

**3.3 Theorem:**

Let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold and  $f : M \circlearrowright$  a  $C^1$ -diffeomorphism. Then the topological entropy of  $f$  is finite with

$$h_{\text{top}}(f) \leq n \log \sup_{p \in M} \|Df^{-1}(p)\|.$$

**3.4 Remarks:**

(i) Since  $h_{\text{top}}(f) = h_{\text{top}}(f^{-1})$ , Ito's estimate also implies that

$$h_{\text{top}}(f) \leq n \log \sup_{p \in M} \|Df(p)\|.$$

(ii) There are earlier estimates of a similar form for the metric entropy of a diffeomorphism (see, e.g., Kushnirenko [41]).

**3.3 Entropy for Group Endomorphisms and Homogeneous Spaces (1971)**

In [8], Bowen defines the topological entropy  $h_{\text{top},d}(f)$  of a uniformly continuous map  $f : X \circlearrowright$  on an arbitrary metric space  $(X, d)$  via  $(n, \varepsilon)$ -spanning and  $(n, \varepsilon)$ -separated sets. In general, this quantity depends on the metric  $d$ . But for  $X$  being compact it coincides with the topological entropy as defined by Adler, Konheim and McAndrew [1].

**3.5 Definition:**

Let  $(X, d)$  be a metric space and  $f : X \rightarrow X$  a uniformly continuous map. For each  $n \in \mathbb{N}$ ,

$$d_{n,f}(x, y) := \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y))$$

defines a metric on  $X$ , topologically equivalent to  $d$ . A set  $E \subset X$  is  $(n, \varepsilon)$ -separated if for all  $x, y \in E$  with  $x \neq y$  it holds that  $d_{n,f}(x, y) > \varepsilon$ . A set  $F \subset X$   $(n, \varepsilon)$ -spans another set  $K \subset X$  if for every  $x \in K$  there is  $y \in F$  with  $d_{n,f}(x, y) \leq \varepsilon$ . For a compact set  $K \subset X$  let  $r_n(\varepsilon, K)$  be the smallest cardinality of a set which  $(n, \varepsilon)$ -spans  $K$ , and let  $s_n(\varepsilon, K)$  be the largest cardinality of an  $(n, \varepsilon)$ -separated set contained in  $K$ . Define

$$\bar{r}_f(\varepsilon, K) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(\varepsilon, K),$$

$$\bar{s}_f(\varepsilon, K) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\varepsilon, K).$$

It holds that

$$h_{\text{top},d}(f, K) := \lim_{\varepsilon \searrow 0} \bar{r}_f(\varepsilon, K) = \lim_{\varepsilon \searrow 0} \bar{s}_f(\varepsilon, K),$$

and one defines

$$h_{\text{top},d}(f) := \sup_{\substack{K \subset X \\ \text{compact}}} h_{\text{top},d}(f, K).$$

Bowen proves the following properties of  $h_{\text{top},d}(f)$ :

**3.6 Theorem:**

(i) If  $d_1$  and  $d_2$  are uniformly equivalent metrics<sup>7</sup> on  $X$ , then

$$h_{\text{top},d_1}(f) = h_{\text{top},d_2}(f).$$

(ii) If  $X$  is compact,  $h_{\text{top},d}(f)$  does not depend on the metric  $d$  and

$$h_{\text{top},d}(f) = h_{\text{top}}(f).$$

(iii) For iterates it holds that

$$h_{\text{top},d}(f^k) = k \cdot h_{\text{top},d}(f) \text{ for all } k \in \mathbb{N}.$$

(iv) Let  $f : X \curvearrowright$  and  $g : Y \curvearrowright$  be uniformly continuous maps on metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , respectively. Define the product metric

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max \{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

on  $X \times Y$ . Then<sup>8</sup>

$$h_{\text{top},d_{X \times Y}}(f \times g) \leq h_{\text{top},d_X}(f) + h_{\text{top},d_Y}(g).$$

Bowen also proves a special case of the *variational principle* for group endomorphisms:

**3.7 Theorem:**

Let  $G$  be a compact metrizable group,  $A : G \curvearrowright$  a surjective endomorphism, and  $\mu$  the normalized Haar measure on  $G$ . Then

$$h_\mu(R_g \circ A) = h_\mu(A) = h_{\text{top},d}(A)$$

for each  $g \in G$ , where  $R_g$  denotes the right translation by  $g$ .

Moreover, he shows the following estimate, which generalizes Ito's estimate from [31]:

<sup>7</sup>Two metrics  $d_1$  and  $d_2$  are called uniformly equivalent if the identity maps  $\text{id} : (X, d_1) \rightarrow (X, d_2)$  and  $\text{id} : (X, d_2) \rightarrow (X, d_1)$  are uniformly continuous.

<sup>8</sup>In the original paper, Bowen stated that equality holds in the product theorem (statement (iv)), but in 1973, an Erratum appeared (see Bowen [10]), where this was corrected.

**3.8 Theorem:**

Let  $f : M \circlearrowright$  be a  $C^1$ -map on an  $n$ -dimensional Riemannian manifold  $(M, g)$ .  
Then

$$h_{\text{top},d}(f) \leq \max \left\{ 0, n \log \sup_{p \in M} \|Df(p)\| \right\},$$

where  $d$  is the Riemannian distance, and the right-hand side may be  $\infty$ .

He also computes the topological entropy of linear maps on  $\mathbb{R}^n$ :

**3.9 Theorem:**

If  $f : \mathbb{R}^n \circlearrowright$  is a linear map and  $d$  is induced by a norm, then

$$h_{\text{top},d}(f) = \sum_{\lambda \in \sigma(f)} \max\{0, d_\lambda \log |\lambda|\},$$

where  $\sigma(f)$  denotes the spectrum of  $f$ , and  $d_\lambda$  is the algebraic multiplicity of the eigenvalue  $\lambda$ .

**3.10 Remark:**

Note that from the above formula it follows that in general the identity  $h_{\text{top},d}(f) = h_{\text{top},d}(f^{-1})$  does not hold if  $f$  is invertible.

**3.11 Corollary:**

If  $f$  is an endomorphism of a Lie group  $G$  and  $d$  is a right-invariant metric on  $G$ , then

$$h_{\text{top},d}(f) = \sum_{\lambda \in \sigma(Df(e))} \max\{0, d_\lambda \log |\lambda|\},$$

where  $e$  denotes the neutral element in  $G$ .

**3.12 Theorem:**

Let  $(X, d_X)$  and  $(Y, d_Y)$  be compact metric spaces,  $T : X \circlearrowright$  and  $S : Y \circlearrowright$  continuous maps, and  $\pi : X \rightarrow Y$  a surjective continuous mapping such that  $\pi \circ T = S \circ \pi$ . Then

$$h_{\text{top},d_X}(T) \leq h_{\text{top},d_Y}(S) + \sup_{y \in Y} h_{\text{top},d_X}(T, \pi^{-1}(y)).$$

**3.13 Remark:**

Note that the above theorem implies that semiconjugate maps have the same topological entropy if the semiconjugacy is finite-to-one.

Finally, Bowen generalizes Ito's result from [30]:

**3.14 Theorem:**

Let  $X$  be a metric space and  $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$  a uniformly continuous semiflow<sup>9</sup> on  $X$ . Then

$$h_{\text{top},d}(\varphi_t) = t \cdot h_{\text{top},d}(\varphi_1) \text{ for all } t \geq 0.$$

**3.15 Remark:**

In Hood [29], Bowen's definition of topological entropy for uniformly continuous maps on metric spaces is generalized to uniformly continuous maps on uniform spaces.

**3.4 Relating Topological Entropy and Measure Entropy (1971)**

In [20], Goodman proves the *variational principle* in the following general form:

**3.16 Theorem:**

Let  $X$  be a compact Hausdorff space and  $f : X \rightarrow X$  a continuous map. Then

$$h_{\text{top}}(f) = \sup_{\mu} h_{\mu}(f),$$

where the supremum is taken over all regular  $f$ -invariant Borel probability measures on  $X$ .

**3.17 Remark:**

Note that the statement of the theorem can be strengthened by taking the supremum only over the ergodic invariant measures.<sup>10</sup> In Misiurewicz [51], an alternative, shorter proof of the variational principle can be found, which is nowadays more popular.

**3.5 On the Relations among various Entropy Characteristics of Dynamical Systems (1971)**

In [16], Dinaburg proves the *variational principle* for homeomorphisms of finite-dimensional spaces:

**3.18 Theorem:**

Let  $X$  be a compact metric space of finite topological dimension and  $f : X \rightarrow X$  a homeomorphism. Then

$$h_{\text{top}}(f) = \sup_{\mu} h_{\mu}(f),$$

<sup>9</sup>The semiflow  $\varphi$  is called uniformly continuous if for all  $t_0 > 0$  and  $\varepsilon > 0$  there is  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $d(\varphi_t(x), \varphi_t(y)) < \varepsilon$  for all  $t \in [0, t_0]$ .

<sup>10</sup>An  $f$ -invariant measure  $\mu$  is called ergodic if  $f^{-1}(A) = A$  for a measurable set  $A$  implies  $\mu(A) = 0$  or  $\mu(A) = 1$ .

where the supremum is taken over all ergodic  $f$ -invariant Borel probability measures on  $X$ .

Amongst others, he also proves the following result:

**3.19 Theorem:**

Let  $(X, d)$  be a compact metric space of finite topological dimension and  $f : X \rightarrow X$  an expansive homeomorphism. Then there exists an  $f$ -invariant Borel probability measure  $\mu$  on  $X$  such that  $h_{\text{top}}(f) = h_{\mu}(f)$ .

In this paper, Dinaburg also introduces the alternative characterization of topological entropy via  $(n, \varepsilon)$ -spanning and  $(n, \varepsilon)$ -separated sets (independently of Bowen). He writes that the idea for this relation stems from Kolmogorov.

### 3.6 Entropy-Expansive Maps (1972)

In [9], Bowen defines the notion of an *entropy-expansive* or  *$h$ -expansive map* (or *homeomorphism*), which generalizes the notion of an *expansive map* (or *homeomorphism*):

**3.20 Definition:**

Let  $f : X \rightarrow X$  be a homeomorphism on a metric space  $(X, d)$ . For all  $x \in X$  and  $\varepsilon > 0$  let

$$\Gamma_{\varepsilon}(x) := \{y \in X : d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \in \mathbb{Z}\}.$$

Then  $f$  is called  *$h$ -expansive* if there exists an  $\varepsilon > 0$  such that  $h_{\text{top}, d}(f, \Gamma_{\varepsilon}(x)) = 0$  for all  $x \in X$ .<sup>11</sup>

Examples for  $h$ -expansive maps are expansive maps, linear maps on Euclidean space, endomorphisms of Lie groups and many more.<sup>12</sup> Bowen proves two results for an entropy-expansive map (or homeomorphism)  $f : X \rightarrow X$  on a compact metric space  $(X, d)$ :

**3.21 Theorem:**

In the definition of topological entropy via  $(n, \varepsilon)$ -spanning sets, one does not have to take the limit for  $\varepsilon \searrow 0$ . Precisely, it holds that

$$h_{\text{top}}(f) = h_{\text{top}}(f, \varepsilon)$$

for  $\varepsilon > 0$  chosen according to the definition of  $h$ -expansivity.

<sup>11</sup>For noninvertible continuous maps  $\mathbb{Z}$  is replaced by  $\mathbb{N}_0$  in the definition of  $\Gamma_{\varepsilon}(x)$ .

<sup>12</sup>In [50], Misiurewicz shows the existence of diffeomorphisms which are not  $h$ -expansive. In the same paper, he also constructs examples of  $C^r$ -maps for finite  $r$ , which have no invariant measure of maximal metric entropy. Using these examples, he is able to show that topological entropy is not an upper semicontinuous function of the diffeomorphism in the  $C^r$ -topology.

**3.22 Theorem:**

Assume that  $X$  has finite topological dimension and  $\mu$  is an  $f$ -invariant Borel probability measure on  $X$ . Then, for the metric entropy of  $f$  with respect to  $\mu$  it holds that

$$h_\mu(f) = h_\mu(f, \mathcal{A}),$$

whenever  $\mathcal{A}$  is a finite measurable partition of  $X$  with sets of diameter at most  $\varepsilon$ .

**3.7 Topological Entropy for Noncompact Sets (1973)**

In [11], Bowen extends the concept of topological entropy. For a continuous map  $f : X \rightarrow X$  on a topological space  $X$  and for any subset  $Y \subset X$  he defines the entropy  $h(f, Y)$  in a way which resembles the definition of Hausdorff dimension. For  $X$  being compact and  $Y = X$  the new entropy coincides with the old one, i.e.,  $h(f, X) = h_{\text{top}}(f)$ . If  $\mu$  is an  $f$ -invariant Borel probability measure and  $\mu(Y) = 1$ , then  $h_\mu(f) \leq h(f, Y)$ . Bowen's definition reads as follows.

**3.23 Definition:**

Let  $X$  be a topological space and  $f : X \rightarrow X$  a continuous map. Let  $Y \subset X$ . If  $\mathcal{A}$  is a finite open cover of  $X$  and  $E \subset X$ , we write  $E \preceq \mathcal{A}$  if  $E$  is contained in an element of  $\mathcal{A}$ , and  $\{E_i\} \preceq \mathcal{A}$  if  $E_i \preceq \mathcal{A}$  for all  $i$ . Let  $n_{f, \mathcal{A}}(E)$  be the biggest nonnegative integer such that

$$f^k E \preceq \mathcal{A} \quad \text{for all } 0 \leq k < n_{f, \mathcal{A}}(E).$$

Put  $n_{f, \mathcal{A}}(E) = 0$  if  $E \not\preceq \mathcal{A}$  and  $n_{f, \mathcal{A}}(E) = \infty$  if  $f^k E \preceq \mathcal{A}$  for all  $k$ . Now put

$$D_{\mathcal{A}}(E) := \exp(-n_{f, \mathcal{A}}(E)), \quad D_{\mathcal{A}}(\mathcal{E}, \lambda) := \sum_{i=1}^{\infty} D_{\mathcal{A}}(E_i)^\lambda$$

for  $\mathcal{E} = \{E_i\}_{i=1}^{\infty}$ ,  $\lambda \in \mathbb{R}$ . Define a measure  $m_{\mathcal{A}, \lambda}$  by

$$m_{\mathcal{A}, \lambda}(Y) := \liminf_{\varepsilon \searrow 0} \left\{ D_{\mathcal{A}}(\mathcal{E}, \lambda) : \bigcup E_i \supset Y \text{ and } D_{\mathcal{A}}(E_i) < \varepsilon \right\}.$$

Notice that  $m_{\mathcal{A}, \lambda}(Y) \leq m_{\mathcal{A}, \lambda'}(Y)$  for  $\lambda > \lambda'$  and  $m_{\mathcal{A}, \lambda}(Y) \notin \{0, \infty\}$  for at most one  $\lambda$ . Define

$$h_{\mathcal{A}}(f, Y) := \inf \{ \lambda : m_{\mathcal{A}, \lambda}(Y) = 0 \}, \quad h(f, Y) := \sup_{\mathcal{A}} h_{\mathcal{A}}(f, Y),$$

where the supremum ranges over all finite open covers of  $X$ .

**3.24 Remark:**

Other papers dealing with extensions of topological entropy to noncompact spaces are Hofer [28], Handel & Kitchens [26], and Patrão [58] (and many more).

### 3.8 Topological Entropy and the First Homology Group (1975)

In [45], Manning proves:

#### 3.25 Theorem:

Let  $f : M \curvearrowright$  be a continuous map on a compact connected smooth manifold  $M$ . Then

$$h_{\text{top}}(f) \geq \log |\lambda|$$

for each eigenvalue  $\lambda$  of the induced map  $f_* : H_1(M, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$  on the first homology group of  $M$  over the reals.

#### 3.26 Remark:

This result is related to *Shub's entropy conjecture* (formulated in [67] by M. Shub) which states that for any  $C^1$ -map  $f : M \curvearrowright$  of a compact smooth manifold  $M$ ,  $h_{\text{top}}(f)$  is bounded from below by the logarithm of the spectral radius of the linear map  $f_*$  induced by  $f$  on the total homology of  $M$  over the reals. In this generality, the conjecture has neither been proved nor disproved until today. Further references about the conjecture are Katok's survey [32] from 1977, and the more recent papers Marzantowicz & Przytycki [46, 47] and Saghin & Xia [66].

### 3.9 Topological Entropy and Degree of Smooth Mappings (1977)

In [54], Misiurewicz and Przytycki prove the following theorem:

#### 3.27 Theorem:

Let  $f : M \curvearrowright$  be a  $C^1$ -map of degree  $\pm N$  on a smooth compact manifold  $M$ . Then

$$h_{\text{top}}(f) \geq \log N.$$

Also this result is related to *Shub's entropy conjecture*, since the degree of a map  $f$  coincides with the spectral radius of the induced map on the top-dimensional homology group  $H_m(M)$ ,  $m = \dim M$ .

The authors also show that the result fails for a  $C^0$ -map.

### 3.10 Characteristic Lyapunov Exponents and Smooth Ergodic Theory (1977)

In [59], Pesin derives a formula for the metric entropy of a  $C^2$ -diffeomorphism preserving a smooth measure:

**3.28 Theorem:**

Let  $M$  be a compact Riemannian manifold and  $f : M \rightarrow M$  a  $C^2$ -diffeomorphism. Let  $\nu$  be a smooth<sup>13</sup>  $f$ -invariant Borel probability measure on  $M$ . For each  $x \in M$  let  $\chi_i(x), i = 1, \dots, s(x)$ , be the Lyapunov exponents at  $x$ , and let  $q_i(x)$  be the multiplicity of  $\chi_i(x)$ . Moreover, let  $k(x)$  be the number of negative Lyapunov exponents at  $x$ . Then

$$h_\nu(f) = - \int \sum_{i=1}^{k(x)} q_i(x) \chi_i(x) d\nu(x).$$

**3.29 Remark:**

Since  $h_\nu(f) = h_\nu(f^{-1})$  and the Lyapunov exponents of  $f^{-1}$  are minus the Lyapunov exponents of  $f$ , Pesin's formula can also be written in the better known form

$$h_\nu(f) = \int \sum_i \max\{0, q_i(x) \chi_i(x)\} d\nu(x).$$

If, in addition, the measure  $\nu$  is ergodic, the integrand is constant almost everywhere, and hence

$$h_\nu(f) = \sum_i \max\{0, q_i(x) \chi_i(x)\}$$

for  $\nu$ -almost all  $x \in M$ .

**3.30 Remark:**

An alternative proof of Pesin's formula can be found in Mañé [43]. In Qian & Zhang [62], Pesin's entropy formula is proved for Axiom A basic sets of  $C^2$ -endomorphisms.

**3.11 An Inequality for the Entropy of Differentiable Maps (1978)**

In [64], Ruelle proves that in Pesin's formula from [59], the inequality " $\leq$ " still holds if  $f$  is only a  $C^1$ -map preserving a (not necessarily smooth) measure:

**3.31 Theorem:**

Let  $M$  be a smooth compact manifold and  $f : M \rightarrow M$  a  $C^1$ -map. Let  $\mu$  be an  $f$ -invariant Borel probability measure on  $M$ . Then

$$h_\mu(f) \leq \int \sum_{\lambda_x} \max\{0, m_x \lambda_x\} d\mu(x),$$

where  $\lambda_x$  are the ( $\mu$ -almost everywhere defined) Lyapunov exponents of  $x$  with associated multiplicities  $m_x$ .

<sup>13</sup>A measure on a smooth manifold is called smooth if it is equivalent to a Riemannian volume measure.

**3.32 Remark:**

Note that the inequality in the above theorem was already proved by Margulis in the volume-preserving case, although he did not publish this result. Sometimes it is called the *Margulis-Ruelle inequality*, or just the *Ruelle-inequality*.

**3.12 Entropy and the Fundamental Group (1978)**

In [12], Bowen proves the following theorems:

**3.33 Theorem:**

Let  $f : M \circlearrowright$  be a continuous map on a compact manifold  $M$ . Then

$$h_{\text{top}}(f) \geq \log \mu(f_*),$$

where  $\mu(f_*)$  is the growth rate of  $f_* : \pi_1(M) \circlearrowright$  (the induced endomorphism on the fundamental group of  $M$ ).<sup>14</sup>

**3.34 Theorem:**

Let  $A : \mathbb{R}^n \circlearrowright$  be linear and  $f : \mathbb{R}^n \circlearrowright$  continuous with  $\|A - f\|_\infty < \infty$ . Then

$$h_{\text{top}}(f) \geq h_{\text{top}}(A),$$

where  $h_{\text{top}}(\cdot) = h_{\text{top},d}(\cdot)$ ,  $d(v, w) = \|v - w\|$ .

In the second theorem, Bowen uses his definition of topological entropy for maps on arbitrary metric spaces, introduced in [8].

**3.13 The Estimation from above for the Topological Entropy of a Diffeomorphism (1979)**

In [36], S. Katok proves the following upper estimate for the topological entropy of a diffeomorphism:

**3.35 Theorem:**

Let  $f : M \circlearrowright$  be a  $C^1$ -diffeomorphism on a compact Riemannian manifold. Then

$$h_{\text{top}}(f) \leq \log \max_{x \in M} \max_{L \subset T_x M} |\det(Df(x)|_L)|,$$

where the inner maximum is taken over all linear subspaces  $L$  of  $T_x M$ .

<sup>14</sup>The growth rate  $\mu(\alpha)$  of an endomorphism  $\alpha : G \rightarrow G$  of a finitely generated group  $G$  is defined as follows: Let  $S = \{g_1, \dots, g_n\}$  be a set of generators for  $G$ . For  $g \in G$  let  $L_S(g)$  be the length of the shortest word in the alphabet  $S \cup S^{-1}$  which represents  $g$ . Then  $\mu(\alpha) := \sup_{g \in G} \limsup_{m \rightarrow \infty} L_S(\alpha^m(g))^{1/m}$ . This quantity does not depend on the set of generators.

**3.36 Corollary:**

Under the assumptions of the above theorem, it holds that

$$h_{\text{top}}(f) \leq \log s(f^*),$$

where  $s(f^*)$  is the spectral radius of the linear map  $f^*$  induced by  $f$  on the space  $\Omega^*(M) = \bigoplus_k \Omega^k(M)$  of differential forms on  $M$ .

**4 The Eighties****4.1 An Upper Estimation for Topological Entropy of Diffeomorphisms (1980)**

In [61], Przytycki proves the following estimate, improving the result of S. Katok in [36]:

**4.1 Theorem:**

Let  $f : M \rightarrow M$  be a  $C^{1+\varepsilon}$ -diffeomorphism on a compact smooth Riemannian manifold  $M$  and let  $\mu$  be the Riemannian volume on  $M$ . Then

$$h_{\text{top}}(f) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \int_M \|Df^n(x)^\wedge\| d\mu(x),$$

where  $Df^n(x)^\wedge$  is the mapping induced by  $Df^n(x)$  between the full exterior algebras of  $T_x M$  and  $T_{f^n(x)} M$ . Geometrically,  $\|Df^n(x)^\wedge\|$  is the maximal volume of the image of an arbitrarily dimensional cube of volume 1 under the differential  $Df^n(x)$ .

**4.2 Remark:**

The right-hand side in the above inequality is independent of the Riemannian metric imposed on  $M$ .

**4.2 Entropy of Piecewise Monotone Mappings (1980)**

In [55], Misiurewicz and Szlenk prove (amongst others) the following formula for the topological entropy of a piecewise monotone map on a compact interval:

**4.3 Theorem:**

Let  $f : I \rightarrow I$  be a continuous, piecewise monotone map on a compact interval  $I \subset \mathbb{R}$ . Let  $c_n$  denote the number of pieces of monotonicity of  $f^n$ . Then

$$h_{\text{top}}(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log c_n.$$

**4.4 Remark:**

There are many other papers and books dealing with dynamics and in particular with topological entropy of interval maps. We only mention a few: Alsedà, Llibre, Misiurewicz [2], Block & Coppel [5], de Melo & van Strien [48], Milnor & Thurston [49], Misiurewicz [52], and L.-S. Young [71].

**4.3 Lyapunov Exponents, Entropy and Periodic Orbits for Diffeomorphisms (1980)**

In [33], A. Katok proves:

**4.5 Theorem:**

Let  $f : M \circlearrowright$  be a  $C^{1+\varepsilon}$ -diffeomorphism on a compact smooth manifold  $M$  and  $\mu$  a Borel probability measure on  $M$  with non-zero Lyapunov exponents. Then

$$h_\mu(f) \leq \max \left\{ 0, \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f) \right\},$$

where  $P_n(f)$  denotes the number of  $n$ -periodic points of  $f$ .

Moreover, he obtains the following corollary:

**4.6 Corollary:**

If  $M$  is two-dimensional, then

$$h_{\text{top}}(f) \leq \max \left\{ 0, \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f) \right\}.$$

**4.7 Remark:**

In this paper, the author also provides an alternative characterization of metric entropy with respect to ergodic measures for homeomorphisms on compact metric spaces via  $(n, \varepsilon)$ -spanning sets.

**4.4 The Metric Entropy of Diffeomorphisms (Part I and II) (1985)**

In [42], for  $C^2$ -diffeomorphisms, Ledrappier and Young give a complete solution to the problem when Pesin's formula from [59] holds for the metric entropy. Moreover, they derive a general formula for the metric entropy. Precisely, they prove the following two results:

**4.8 Theorem:**

Let  $M$  be a smooth compact manifold,  $f : M \circlearrowright$  a  $C^2$ -diffeomorphism and  $\mu$  an  $f$ -invariant probability Borel measure on  $M$ . Then the equality

$$h_\mu(f) = \int \sum_i \max\{0, m_i(x)\lambda_i(x)\} d\mu(x)$$

is equivalent to  $\mu$  having absolutely continuous conditional measures on unstable manifolds.<sup>15</sup>

#### 4.9 Theorem:

Under the assumptions of the first theorem, in general, the formula

$$h_\mu(f) = \int \sum_i \max\{0, \gamma_i(x)\lambda_i(x)\} d\mu(x)$$

holds, where  $\gamma_i(x) \in [0, m_x]$  is the “dimension of  $\mu$  in the direction of the subspace  $E_i(x)$ ”.<sup>16</sup>

### 4.5 Volume Growth and Entropy (1987)

In [69, 70], Yomdin proves *Shub’s entropy conjecture* for maps of class  $C^\infty$ :

#### 4.10 Theorem:

Let  $M$  be a compact  $m$ -dimensional smooth manifold and  $f : M \rightarrow M$  a  $C^\infty$ -map. For  $l = 0, 1, \dots, m$  let  $S_l(f)$  be the logarithm of the spectral radius of  $f_* : H_l(M, \mathbb{R}) \rightarrow H_l(M, \mathbb{R})$  and let  $S(f) := \max_l S_l(f)$ . Then

$$h_{\text{top}}(f) \geq S(f).$$

In order to relate Yomdin’s work to that of Newhouse, we explain his proof more precisely: Define

$$R(f) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \max_{x \in M} \|Df^n(x)\|.$$

For a  $C^k$ -map  $\sigma : Q^l \rightarrow M$ ,  $Q^l = [0, 1]^l$ , let

$$v(\sigma) := \int_{Q^l} \bar{v}(d\sigma),$$

where  $\bar{v}(d\sigma)$  is the volume form on  $Q^l$  induced by  $\sigma$  from a given Riemannian metric on  $M$ . Let  $\Sigma(k, l)$  be the set of all  $C^k$ -mappings  $\sigma : Q^l \rightarrow M$ . Moreover, let

$$v(f, \sigma, n) := v(f^n \circ \sigma).$$

<sup>15</sup> $\mu$  has absolutely continuous conditional measures on unstable manifolds if for every measurable partition  $\xi$  subordinate to  $W^u$ ,  $\mu_x^\xi$  is absolutely continuous with respect to the induced Riemannian measure on the unstable manifold  $W^u(x)$ , for almost every  $x$ . Here, a measurable partition  $\xi$  is said to be subordinate to  $W^u$  if for almost every  $x$ ,  $\xi(x) \subset W^u(x)$  and  $\xi(x)$  contains a neighborhood of  $x$  open in the submanifold topology of  $W^u(x)$ , where  $\xi(x)$  is the element of  $\xi$  containing  $x$ . Moreover,  $\mu_x^\xi$  is a conditional measure on  $\xi(x)$ .

<sup>16</sup>The quite technical definition of this will not be made precise here.

Now, for  $k \geq 1$  and  $l \leq m = \dim M$ , define

$$\begin{aligned} v_{l,k}(f) &:= \sup_{\sigma \in \Sigma(l,k)} \limsup_{n \rightarrow \infty} \frac{1}{n} \log v(f, \sigma, n), \\ v_k(f) &:= \max_l v_{l,k}(f), \\ v(f) &:= v_\infty(f). \end{aligned}$$

For  $f$  being  $C^{1+\varepsilon}$ ,  $\varepsilon > 0$ , Newhouse [56] shows that

$$h_{\text{top}}(f) \leq v(f).$$

Yomdin proves the converse inequality. More precisely, he shows that

$$v_{l,k}(f) \leq h_{\text{top}}(f) + \frac{2l}{k} R(f)$$

for  $k = 1, \dots, \infty$  and  $l \leq m$ . Taking the maximum over  $l$  and letting  $k$  go to infinity, the inequality  $v(f) \leq h_{\text{top}}(f)$  follows in case  $f$  is of class  $C^\infty$ . The proof of the above theorem then follows from the inequality

$$S(f) \leq h_{\text{top}}(f) + \frac{2m}{k} R(f).$$

#### 4.11 Remark:

The results of Yomdin also imply that the function  $h_{\text{top}} : C^\infty(M) \rightarrow \mathbb{R}$  is upper semicontinuous. In [57], Newhouse gives a different proof for this fact.

## 4.6 Entropy and Volume (1988)

In order to formulate Newhouse's main result in [56], we need the following definition:

#### 4.12 Definition:

Let  $(M, g)$  be a Riemannian manifold of class  $C^{1+\alpha}$  and  $f : M \rightarrow M$  a  $C^{1+\alpha}$ -map. Let  $\Omega$  be a compact  $f$ -invariant set. For  $1 \leq k \leq \dim M$ , let  $D^k$  be the unit disc in  $\mathbb{R}^k$ . A smooth  $k$ -disc in  $M$  is a  $C^{1+\alpha}$ -map  $\gamma : D^k \rightarrow M$ . The Riemannian metric  $g$  induces a norm  $\|\cdot\|_k$  on each exterior power  $\Lambda^k T_x M$  of the tangent space  $T_x M$ . Define the  $k$ -volume of  $\gamma$  by

$$\|\gamma\| := \int_{D^k} \left\| \Lambda^k D\gamma(x) \right\|_k d\lambda(x),$$

where  $\Lambda^k D\gamma(x) : \Lambda^k T_x D^k \rightarrow \Lambda^k T_{\gamma(x)} M$  is the linear map on the  $k$ -th exterior power induced by the derivative  $D\gamma(x)$ , and  $d\lambda$  is the standard volume form on  $D^k$ . Let  $\mathcal{A}$  be a collection of  $C^{1+\alpha}$ -discs in  $M$  whose dimensions vary from 1 through  $\dim M$ . Let  $G^k(\Omega)$  be the Grassmann bundle

of  $k$ -planes over  $\Omega$  and let  $\bigcup_k G^k(\Omega)$  be the disjoint union. Assume that  $M \subset \mathbb{R}^N$  for large  $N$ , so that for  $x, y \in M$  it is meaningful to write  $\|y - x\|$ . For  $\gamma \in \mathcal{A}$ ,  $\gamma : D^k \rightarrow M$ , let

$$\text{Lip}_\alpha(\gamma) := \sup_{\substack{x \neq y \\ x, y \in D^k}} \frac{\|D\gamma(y) - D\gamma(x)\|}{\|y - x\|^\alpha}.$$

Then  $\mathcal{A}$  is said to be ample for  $\Omega$  if there exists  $K > 0$  such that

- (i)  $\inf_{\|v\|=1} \|D\gamma(0)v\| \geq K^{-1}$  for  $\gamma \in \mathcal{A}$ ,
- (ii)  $\text{Lip}_\alpha(\gamma) \leq K$  for  $\gamma \in \mathcal{A}$ ,

(iii)  $\bigcup_{\gamma \in \mathcal{A}} \text{im}(D\gamma(0))$  is dense in  $\bigcup_k G^k(\Omega)$ .  
Let  $V$  be a compact neighborhood of  $\Omega$ . For  $n \in \mathbb{N}$  let  $W^s(n, V) = \bigcup_{0 \leq j < n} f^{-j}(V)$ . For a  $C^{1+\alpha}$ -disc  $\gamma : D \rightarrow V$  define

$$G(\gamma, f, V) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log^+ \left( \left\| f^{n-1} \circ \gamma|_{\gamma^{-1}(W^s(n, V))} \right\| \right),$$

where  $\log^+(x) = \max\{0, \log(x)\}$ .

Newhouse's main result reads as follows:

#### 4.13 Theorem:

Let  $(M, g)$  be a Riemannian manifold of class  $C^2$  and  $f : M \rightarrow M$  a  $C^{1+\alpha}$ -map. Let  $\Omega$  be a compact  $f$ -invariant set,  $U$  a compact neighborhood of  $\Omega$  in  $M$ , and  $\mathcal{A}$  an ample family of smooth discs for  $\Omega$ . Then

$$h_{\text{top}}(f|_\Omega) \leq \sup_{\gamma \in \mathcal{A}} G(\gamma, f, U).$$

#### 4.14 Remark:

Newhouse also proves a special version of the above theorem for holomorphic maps on complex Hermitian manifolds, which allows him to derive relations between the topological entropy and the degree, both for polynomial maps on compact subsets of  $\mathbb{R}^n$ , and for holomorphic maps on complex projective space. These relations have also been shown by Gromov [24].

## 4.7 Continuity Properties of Entropy (1989)

In [57], Newhouse proves the following theorem:

#### 4.15 Theorem:

Let  $M$  be a compact smooth manifold. Then for every  $C^\infty$ -map  $f : M \rightarrow M$  the function  $\mu \mapsto h_\mu(f)$  is upper semicontinuous. Moreover, the mapping  $f \mapsto h_{\text{top}}(f)$  from  $C^\infty(M)$  to  $\mathbb{R}$  is upper semicontinuous.

The proof in particular uses the results of Yomdin [69]. Moreover, Newhouse concludes lower semicontinuity for surface diffeomorphisms from a result of Katok [33]. Together with the above theorem this implies the following corollary.

**4.16 Corollary:**

*The map  $f \mapsto h_{\text{top}}(f)$  is continuous for  $C^\infty$ -diffeomorphisms of two-dimensional compact manifolds.*

**4.17 Remark:**

From Newhouse's theorem it follows that every  $C^\infty$ -map has ergodic invariant measures of maximal entropy, which is not true for maps of less smoothness (see Misiurewicz [50]).

**4.8 Expansiveness, Hyperbolicity and Hausdorff Dimension (1989)**

In [19], Fathi proves the following theorems:

**4.18 Theorem:**

*Let  $f : M \circlearrowleft$  be a  $C^1$ -diffeomorphism on a Riemannian manifold  $M$  and  $K \subset M$  a compact hyperbolic set with respect to  $f$ .<sup>17</sup> Define*

$$\lambda := \max \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \max_{x \in K} \|Df^n(x)|_{E^s}\|, \lim_{n \rightarrow \infty} \frac{1}{n} \log \max_{x \in K} \|Df^{-n}(x)|_{E^u}\| \right\}.$$

*Then it holds that*

$$\dim_H(K) \leq \overline{\dim}_B(K) \leq 2 \frac{h_{\text{top}}(f|_K)}{-\lambda}.$$

**4.19 Theorem:**

*Let  $\varphi : \mathbb{R} \times M \rightarrow M$  be a  $C^1$ -flow on a Riemannian manifold  $M$  and  $K \subset M$  a compact hyperbolic set for  $\varphi$ . Define*

$$\lambda := \max \left\{ \lim_{t \rightarrow \infty} \frac{1}{t} \log \max_{x \in K} \|D\varphi_t(x)|_{E^s}\|, \lim_{t \rightarrow \infty} \frac{1}{t} \log \max_{x \in K} \|D\varphi_{-t}(x)|_{E^u}\| \right\}.$$

*Then it holds that*

$$\dim_H(K) \leq \overline{\dim}_B(K) \leq 2 \frac{h_{\text{top}}(\varphi|_K)}{-\lambda} + 1.$$

---

<sup>17</sup>A compact set  $K$  is *hyperbolic* with respect to  $f$  if it is  $f$ -invariant and if there exists an equivariant tangent bundle splitting  $T_K M = E^s \oplus E^u$  such that  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \max_{x \in K} \|Df^n|_{E^s}\| < 0$  and  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \max_{x \in K} \|Df^{-n}|_{E^u}\| < 0$ .

## 5 The Nineties

### 5.1 Pseudo-Orbits and Topological Entropy (1990)

In [4], Barge and Swanson relate topological entropy to growth rates of pseudo-orbits.

#### 5.1 Definition:

Let  $f : X \circlearrowleft$  be a continuous map on a compact metric space  $(X, d)$ . A collection  $E$  of  $\alpha$ -pseudo-orbits<sup>18</sup> of  $f$  is  $(n, \varepsilon)$ -separated if, for each  $(x_i), (y_i) \in E$ ,  $(x_i) \neq (y_i)$ , there is  $k \in \{0, 1, \dots, n-1\}$  for which  $d(x_k, y_k) > \varepsilon$ . Let  $S(n, \varepsilon, \alpha)$  denote the maximal cardinality of an  $(n, \varepsilon)$ -separated set of  $\alpha$ -pseudo-orbits. The pseudo-entropy of  $f$  is defined by

$$h_\psi(f) := \lim_{\varepsilon \searrow 0} \lim_{\alpha \searrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log S(n, \varepsilon, \alpha).$$

Let  $E$  denote a set of  $(n, \varepsilon)$ -separated periodic  $\alpha$ -pseudo-orbits of period  $n$ . Thus, if  $(x_i), (y_i) \in E$ ,  $(x_i) \neq (y_i)$ , then  $(x_i)$  and  $(y_i)$  are periodic of period  $n$  and  $d(x_k, y_k) > \varepsilon$  for some  $k \in \{0, 1, \dots, n-1\}$ . Let  $P(n, \varepsilon, \alpha)$  denote the maximal cardinality of such a set, and define

$$H_\psi(f) := \lim_{\varepsilon \searrow 0} \lim_{\alpha \searrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log P(n, \varepsilon, \alpha).$$

#### 5.2 Theorem:

It holds that  $h_{\text{top}}(f) = h_\psi(f) = H_\psi(f)$ .

#### 5.3 Corollary:

Let  $f_n : X \circlearrowleft$  be a sequence of continuous maps which converges uniformly to  $f$ . Then

$$\lim_{\varepsilon \searrow 0} \limsup_{n \rightarrow \infty} h_{\text{top}}(f_n, \varepsilon) \leq h_{\text{top}}(f).$$

#### 5.4 Remark:

The equality  $h_{\text{top}}(f) = h_\psi(f)$  has also been proved by Misiurewicz [53].

## 5.2 Topological Entropy of Nonautonomous Dynamical Systems (1996)

In [38], Kolyada and Snoha extend the concept of topological entropy to nonautonomous dynamical systems given by sequences of maps on a compact topological space:

<sup>18</sup>An  $\alpha$ -pseudo-orbit of  $f$  is a (finite or infinite) sequence  $(x_n)$  in  $X$  such that  $d(f(x_n), x_{n+1}) \leq \alpha$  for all  $n$ .

**5.5 Definition:**

Let  $X$  be a compact topological space and  $f_{1,\infty} := \{f_i\}_{i=1}^\infty$  a sequence of continuous maps  $f_i : X \rightarrow X$ . For any  $i, n \in \mathbb{N}$  let

$$f_i^0 = \text{id}_X, \quad f_i^n := f_{i+(n-1)} \circ \dots \circ f_{i+1} \circ f_i.$$

For any open cover  $\mathcal{A}$  of  $X$  let

$$\mathcal{A}_i^n := \bigvee_{j=0}^{n-1} f_i^{-j}(\mathcal{A}),$$

and let  $\mathcal{N}(\cdot)$  denote the minimal cardinality of a subcover. Then define the topological entropy of the sequence of maps  $f_{1,\infty}$  on the cover  $\mathcal{A}$  by

$$h_{\text{top}}(f_{1,\infty}, \mathcal{A}) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N}(\mathcal{A}_1^n).$$

The topological entropy of the sequence of maps  $f_{1,\infty}$  is then defined by

$$h_{\text{top}}(f_{1,\infty}) := \sup_{\mathcal{A}} h_{\text{top}}(f_{1,\infty}, \mathcal{A}),$$

where the supremum is taken over all open covers  $\mathcal{A}$  of  $X$ .

Obviously, for an autonomous system this definition coincides with the original one by Adler, Konheim and McAndrew [1]. The authors also provide equivalent Bowen-like definitions of  $h_{\text{top}}(f_{1,\infty})$  via separated and spanning sets, and they show that the topological entropy for nonautonomous systems shares several properties with the “autonomous entropy”. In particular,  $h_{\text{top}}(f_{1,\infty})$  is an invariant under equiconjugacy.<sup>19</sup> Moreover, they prove the following nonautonomous version of Bowen’s theorem stating that the entropy of a map equals the entropy of the restriction to the nonwandering set:

**5.6 Theorem:**

Let  $f_{1,\infty}$  be a sequence of equicontinuous selfmaps of a compact metric space  $X$ . Then  $h_{\text{top}}(f_{1,\infty}) = h_{\text{top}}(f_{1,\infty}; \Omega(f_{1,\infty}))$ , where  $\Omega(f_{1,\infty})$  denotes the non-wandering set of  $f_{1,\infty}$ , defined as follows: A point  $x \in X$  is said to be non-wandering if for every neighborhood  $U(x)$  of  $x$  there are positive integers  $n$  and  $k$  with  $f_n^k(U(x)) \cap U(x) \neq \emptyset$ . Then  $\Omega(f_{1,\infty})$  is the set of all non-wandering points.

<sup>19</sup>Two nonautonomous dynamical systems  $f_{1,\infty}$  and  $g_{1,\infty}$  on spaces  $X$  and  $Y$  are called topologically equiconjugate if there is an equicontinuous sequence  $(\pi_i)_{i \geq 1}$  of homeomorphisms from  $X$  to  $Y$  such that also the sequence  $(\pi_i^{-1})_{i \geq 1}$  is equicontinuous and  $\pi_{i+1} \circ f_i = g_i \circ \pi_i$  for all  $i \geq 1$ .

Furthermore, the authors discover the following property of the classical topological entropy, which was also proved independently in [14]:

**5.7 Theorem:**

Let  $X$  be a compact topological space and  $f, g : X \rightarrow X$  continuous maps. Then

$$h_{\text{top}}(f \circ g) = h_{\text{top}}(g \circ f).$$

**5.8 Remark:**

The approach of Kolyada and Snoha has been further developed, in particular, in Kolyada & Misiurewicz & Snoha [39], Zhang & Chen [72] and Zhu & Zhang & He [73]. An analogous notion of metric entropy for nonautonomous systems has been established in Kawan [37].

**5.3 An Integral Formula for Topological Entropy of  $C^\infty$ -maps (1998)**

In [40], Kozlovski proves that Przytycki's upper estimate for  $C^{1+\varepsilon}$ -diffeomorphisms in [61] becomes an equality for  $C^\infty$ -maps:

**5.9 Theorem:**

Let  $f : M \rightarrow M$  be a  $C^\infty$ -map on a compact smooth Riemannian manifold  $M$  and let  $\mu$  be the Riemannian volume on  $M$ . Then

$$h_{\text{top}}(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int_M \|Df^n(x)^\wedge\| d\mu(x),$$

where  $Df^n(x)^\wedge$  denotes the induced mapping between the full exterior algebras of the tangent spaces  $T_x M$  and  $T_{f^n(x)} M$ .

Kozlovski's formula in general does not hold for maps of less smoothness, as was already shown by Misiurewicz. Kozlovski's proof is partially based on the results of Yomdin [69].

**5.4 Some Relations between Hausdorff-dimensions and Entropies (1998)**

In [13], Dai, Zhou and Geng prove the following result, which generalizes and improves Ito's early estimate in [31]:

**5.10 Theorem:**

Let  $(X, d)$  be a compact metric space and  $f : X \rightarrow X$  a continuous map. Define the local Lipschitz constant of  $f$  with respect to  $d$  by

$$L(f, d) := \lim_{\varepsilon \searrow 0} \sup_{\substack{d(x,y) < \varepsilon \\ x \neq y}} \frac{d(f(x), f(y))}{d(x, y)}.$$

Then it holds that

$$h_{\text{top}}(f) \leq \max\{0, \log L(f, d)\} \cdot \dim_H(X, d),$$

where  $\dim_H(X, d)$  denotes the Hausdorff dimension of  $(X, d)$ .

### 5.5 Lyapunov's Direct Method in Estimates of Topological Entropy (1998)

In [6], Boichenko and Leonov prove some estimates for topological entropy similar to those of Ito [31], Dai et al. [13], and Eden et al. [18]:

#### 5.11 Theorem:

Let  $(X, d)$  be a compact metric space,  $f : X \rightarrow X$  a continuous map, and  $p : X \times X \rightarrow (0, \infty)$  a positive continuous function. Let  $d'$  be another metric on  $X$  which is metrically equivalent to  $d$ . Set

$$k_j := \limsup_{\varepsilon \searrow 0} \sup_{\substack{d'(x, y) < \varepsilon \\ x \neq y}} \left[ \frac{d'(f^j(x), f^j(y))}{d'(x, y)} \cdot \frac{p(f^j(x), f^j(y))}{p(x, y)} \right], \quad j \in \mathbb{N},$$

$$k := \inf_{j \in \mathbb{N}} k_j^{1/j}.$$

If  $k < \infty$  and  $\underline{\dim}_F(X) < \infty$ , then

$$h_{\text{top}}(f) \leq \max\{0, \log k\} \cdot \underline{\dim}_F(X),$$

where  $\underline{\dim}_F(X)$  denotes the lower box dimension of  $X$ .

#### 5.12 Corollary:

Let  $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$  be a continuous semiflow on a compact metric space  $(X, d)$  such that  $\varphi_t$  is Lipschitz continuous and the Lipschitz constants are bounded on an interval  $[0, t_0]$ . Then

$$h_{\text{top}}(\varphi_1) \leq \max\{0, \nu\} \cdot \dim_F(X)$$

with

$$\nu = \lim_{t \rightarrow \infty} \frac{1}{t} \log \inf \{ \tilde{\nu} > 0 : d(\varphi_t(x), \varphi_t(y)) \leq \tilde{\nu} d(x, y) \forall x, y \in X \}.$$

#### 5.13 Theorem:

Consider an ordinary differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

with  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  of class  $C^1$ . Let  $K \subset \mathbb{R}^n$  be a compact and convex invariant set, i.e.,  $\varphi_t(K) = K$  for all  $t \in \mathbb{R}$ , where  $(\varphi_t)$  denotes the corresponding flow. Let  $v : K \rightarrow \mathbb{R}$  be of class  $C^1$ . Then

$$h_{\text{top}}(\varphi_1, K) \leq \max\{0, \tilde{k}\} \cdot \dim_F(K)$$

with

$$\tilde{k} = \max_{x \in K} [\gamma(Df(x)) + \dot{v}(x)],$$

where  $\gamma(\cdot)$  denotes the logarithmic norm defined by

$$\gamma(A) = \lim_{\varepsilon \searrow 0} \frac{\|I + \varepsilon A\| - 1}{\varepsilon} \text{ for all } A \in \mathbb{R}^{n \times n},$$

and  $\dot{v}(x) = \langle \nabla v(x), f(x) \rangle$ .

#### 5.14 Theorem:

With the assumptions of the preceding theorem, it holds that

$$h_{\text{top}}(\varphi_1, K) \leq \max\{0, \tilde{k}\} \cdot \dim_F(K)$$

with

$$\tilde{k} = \max_{x \in K} [\lambda_1(x) + \dot{v}(x)],$$

where  $\lambda_1(x)$  is the greatest eigenvalue of the symmetric matrix  $\frac{1}{2}(Df(x) + Df(x)^T)$ .

## 6 The New Millenium

### 6.1 Axiomatic Definition of the Topological Entropy on the Interval (2003)

In [3], Alsedà, Kolyada, Llibre and Snoha present two different sets of axioms which characterize the topological entropy function

$$f \mapsto h_{\text{top}}(f), \quad C(I) \rightarrow [0, \infty],$$

for continuous maps on a compact interval  $I$ . To formulate the main theorems, we need some definitions:

#### 6.1 Definition:

Let  $f, g \in C(I)$ . We say that  $g$  is obtained from  $f$  by pouring water and we write  $g \in \text{PW}(f)$ , if there exists an open set  $G \subset I$  (in the relative topology) such that  $g$  is constant on each component of  $G$  and  $g(x) = f(x)$  for all  $x \in I \setminus G$ . For  $\lambda > 0$  we define

$$\mathcal{CS}_\lambda := \{f \in C(I) : f \text{ is piecewise linear with slopes either } \lambda \text{ or } -\lambda\}.$$

Let  $f, g \in C(I)$ . Recall that  $g$  is a factor of  $f$  or, equivalently,  $f$  is semiconjugate to  $g$ , if there is a surjective map  $\phi \in C(I)$  such that  $\phi \circ f = g \circ \phi$ . If, additionally,  $\phi$  is non-decreasing, we say that  $g$  is a strong factor of  $f$ . The class of all strong factors of  $f \in C(I)$  is denoted by  $\text{SF}(f)$ . A set  $P \subset I$  is

called weakly  $f$ -invariant if  $f(P) \subset P$ . Let  $P$  be a finite subset of  $I$ . A map  $f \in C(I)$  will be called  $P$ -monotone (respectively  $P$ -linear), if it is constant on  $[0, \min P]$  and  $[\max P, 1]$ , and  $f$  is (not necessarily strictly) monotone (respectively, affine) on the closure of each connected component of  $I \setminus P$ .

### 6.2 Theorem:

Let  $Ax : C(I) \rightarrow [0, \infty]$  satisfy the following properties:

- (i)  $Ax$  is lower semicontinuous.
- (ii)  $Ax(g) \leq Ax(f)$  if  $g \in PW(f)$ .
- (iii)  $Ax(g) \leq Ax(f)$  if  $g \in SF(f)$ .
- (iv) If  $f$  is a  $P$ -linear map, where  $P$  is a weakly  $f$ -invariant set and  $Ax(f) > 0$ , then  $f$  has a piecewise linear (not necessarily strong) factor  $g \in \mathcal{CS}_\lambda$  for some  $\lambda$  such that  $Ax(f) = Ax(g)$ .
- (v)  $Ax(f) \leq \log \lambda$ , whenever  $f \in \mathcal{CS}_\lambda$  is a  $P$ -linear map, where  $P$  is a weakly  $f$ -invariant set and  $\lambda \geq 1$ .
- (vi)  $Ax(f) \geq \log \lambda$ , whenever  $f \in \mathcal{CS}_\lambda$  is a  $P$ -linear map, where  $P$  is a periodic orbit of  $f$  and  $\lambda > 1$ .

Then  $Ax = h_{\text{top}}$ .

### 6.3 Theorem:

Let  $Ax : C(I) \rightarrow [0, \infty]$  satisfy the following properties:

- (i)  $Ax$  is lower semicontinuous.
- (ii)  $Ax(g) \leq Ax(f)$  if  $g \in PW(f)$ .
- (iii)  $Ax(g) \leq Ax(f)$  if  $g \in SF(f)$ .
- (iv) If  $f$  is a  $P$ -linear map, where  $P$  is a weakly  $f$ -invariant set and  $Ax(f) > 0$ , then  $f$  has a piecewise linear (not necessarily strong) factor  $g \in \mathcal{CS}_\lambda$  for some  $\lambda$  such that  $Ax(f) = Ax(g)$ .
- (v)  $Ax(f \circ g) = Ax(f) + Ax(g)$ , whenever  $f \in \mathcal{CS}_{\lambda_1}$  and  $g \in \mathcal{CS}_{\lambda_2}$  with  $\lambda_1, \lambda_2 \geq 1$ .
- (vi)  $Ax(\tau) = \log 2$ , where  $\tau$  is the standard tent map  $\tau(x) = 1 - |2x - 1|$ .
- (vii)  $Ax(f) = 0$ , whenever  $f \in \mathcal{CS}_1$  has only fixed points, but no periodic points of higher period.

Then  $Ax = h_{\text{top}}$ .

## 6.2 Topological Entropy for Nonuniformly Continuous Maps (2008)

In the paper [27], the authors, Hasselblatt, Nitecki and Propp, study and compare three different extensions (from the literature) of the standard definition of topological entropy for maps on not necessarily compact spaces:

The *Bowen-Compacta-Entropy* (see [8]), the *Friedland Entropy*, and the *Bowen-Dinaburg-Entropy*. In particular, they consider the case of nonuniformly continuous maps.

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